Journal of the ACS, Vol. 8, May 2014

Hyers-Ulam-Rassias Stability For Volterra Integral Equations on Time Scales

Nesreen A.Yaseen
Department of information systems,
Higher institute for computers science
El-Shoruk academy
yehya_73@yahoo.com

Abstract

This paper introduce the Hyers-Ulam-Rassias stability for the Volterra integral equation on time scales of the form

$$y(t) = \int_{t_0}^{t} f\big(t, s, y(s\,)\big)\,\Delta s, \qquad s, t \in \mathbb{T} \ ,$$

where $y \in C_{rd}(\mathbb{T}, \mathbb{X})$, and $\mathbb{X}$ is a Banach space

Keywords: Dynamic equations on time scales, Hyers-Ulam- Rassias stability, Volterra Integral equations on time scales.

1 . Introduction

A time scale $\mathbb{T}$ is an arbitrary non-empty closed subset of the set of real number. The object of the theory of dynamic equations on time scales is to unify continuous and discrete calculus which was introduced by Stefan Hilger in his Ph. D. in 1988 [16]. The theory presented a structure where, once a result is established for general time scale, special cases include a result for differential equations and a result for difference equations. A great deal of work has been done since 1988 unifying the theory of differential equations and the theory of difference equations by establishing the corresponding results in time scale setting. Recent two books of Bohner and Paterson [6,7] provide both an excellent introduction to the subject and up to-date coverage of much of the theory . Studying the theory of dynamic equations on time scales has attracted many authors in the last few years, because of the wide applications in engineering, industry, biology, economics and other field, it appears to be advantageous to model certain processes by employing a suitable combination of both differential equations and difference equations at different stages in the process under consideration. Also, the stability analysis of dynamic equations has become an important topic both theoretically and practically because dynamic equations occur in many areas such as mechanics, physics, and economics.

This paper focuses on the Hyers-Ulam-Rassias stability of Volterra Integral equations on time scales . It well known that the idea of Hyers-Ulam and Hyers-Ulam-Rassias stability was presented after many years of introducing Ulam [22] his problem of stability in 1940 which solved by Hyers [17], and the result of Hyers was generalized by Rassias [20]. Alsina and Ger [15] were the first authors who investigated the Hyers-Ulam stability of a differential equation. Also of interest, that many of articles were edited by Rassias [21], dealing with Ulam, Hyers-Ulam and Hyers-Ulam-Rassias stability . Many papers introduced the Hyers-Ulam and Hyers-Ulam-Rassias stability for differential equations and integral equations[10,18]. On the other hand side the papers which were presented the Hyers- Ulam and Hyers-Ulam-Rassias stability of dynamic equations are still very few, may be except the studies which were presented in the papers [3,4,5].

Although there are numerous publications for different types of equations, there are very few results on the study of these kinds of stabilities for integral equations .

Volterra integral equations have been studied in a quite extensive way since the four fundamental papers of Vito Volterra in 1896, and specially since 1913 when Volterra's book Le¸ cons sur les´Equations Int´egrales et les´Equations Inte'grodiffe'rentielles appeared. Part of this interest arises directly from the applications where this kind of equations appears. This is the case in elasticity, semiconductors, scattering theory, seismology, heat conduction, fluid flow, chemical reactions, population dynamics, etc. (see [9,12, 19]).

Despite the large amount of works on Volterra integral equations, up to our knowledge only the work [18] studies conditions which ensure Hyers Ulam–Rassias stability and Hyers–Ulam stability of a certain type of Volterra integral equations (see12,13,14) and [18]). In [10] Castro presented a Hyers–Ulam–Rassias stability study for the nonlinear Volterra integral equations of the form

$$y(x) = \int_a^x f\big(x, \tau, y(\tau)\big)\, d\tau \quad (-\infty < a \leq x \leq b < +\infty), \qquad (1.1)$$

where a and b are fixed real numbers and f is a continuous function. He noted that this class of functional equations is more global than the one considered in [18]. He proved the Hyers–Ulam stability and the Hyers–Ulam-Rassias stability of the Volterra integral equation (1.1).

This paper focuses on the Volterra integral equations on time scales.

In [11] C. Constanda presented both Hyers–Ulam and Hyers–Ulam–Rassias stability study for the delay Volterra-type integral equations

In 2011, by using a fixed point method, Mohamed Akkouchi [1] established the Hyers–Ulam stability and the Hyers–Ulam–Rassias stability for a general class of nonlinear Volterra integral equations in Banach spaces.

The field of studying Hyers-Ulam and Hyers-Ulam-Rassias stability of dynamic equations is still limited, specially up to our knowledge there is no paper presented to study the Hyers-Ulam-Rassias stability of Volterra integral dynamic equations .

2. Preliminaries

We need the following definitions and notations from [8] in proving our main results in Section 3.

Definition 2.1. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.

Definition 2.2. The mappings $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by $\sigma(t) = inf\{s \in \mathbb{T}: s > t\}$, and $\rho(t) = sup\{s \in \mathbb{T}: s < t\}$ are called the jump operators.

Definition 2.3. A point $t \in \mathbb{T}$ is said to be ***right–dense*** if $\sigma(t) = t$, ***right-scattered*** if $\sigma(t) > t$, ***left-dense*** *if* $\rho(t) = t$, ***left-scattered*** *if* $\rho(t) < t$, ***isolated*** if $\rho(t) < t < \sigma(t)$, and dense if $\rho(t) = t = \sigma(t)$.

Definition 2.4. Let $t \in \mathbb{T}$. The graininess function

$\mu: \mathbb{T} \to [0, \infty[$ is defined by $\mu(t) = \sigma(t) - t$.

Definition 2.5. A function $f: \mathbb{T} \to \mathbb{X}$ is called ***rd-continuous*** provided

(i) f is continuous at every right-dense point;

(ii) $lim_{s \to t^-} f(s)$ exists (finite) at left-dense points in $\mathbb{T}$.

The set of rd-continuous functions $f: \mathbb{T} \to \mathbb{X}$ will be denoted by $C_{rd}(\mathbb{T}, \mathbb{X})$.

Definition 2.6. (The Delta Derivative). A function $f: \mathbb{T} \to \mathbb{X}$ is called Δ-***differentiable*** at $t \in \mathbb{T}^k$ if there exists an element $f^{\Delta}(t) \in \mathbb{X}$ such that for any $\varepsilon > 0$ there is $\delta > 0$ such that:

$$\left\| \left[f\big(\sigma(t)\big) - f(s) \right] - f^{\Delta}(t)[\sigma(t) - s] \right\| \le \varepsilon |\sigma(t) - s|, s \in (t - \delta, t + \delta) \cap \mathbb{T}.$$

In this case $f^{\Delta}(t)$ is called the ***delta derivative*** of f at t, provided it exists and we have

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f\big(\sigma(t)\big) - f(s)}{\sigma(t) - s}. \qquad (2.1)$$

If $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$, we say that f is delta differentiable on $\mathbb{T}^k$. Here

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & if \sup \mathbb{T} < \infty \\ \mathbb{T} & if \sup \mathbb{T} = \infty. \end{cases}$$

We also denote by

$$C_{rd}^{n}(\mathbb{T}, \mathbb{X}) = \left\{ x \in C_{rd}(\mathbb{T}, \mathbb{X}) : x^{\Delta}, \dots, x^{\Delta^n} \text{exist and belong to } C_{rd}(\mathbb{T}, \mathbb{X}) \right\}.$$

Definition2.7. A function $f: \mathbb{T} \to \mathbb{R}$ is called ***regulated*** provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 2.8. Let $f: \mathbb{T} \to \mathbb{X}$ be regulated function. Any function F which satisfies $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$, is called a pre-antiderivative of f. We define the ***indefinite integral*** of a regulated function f by

$$\int f(t) \Delta t = F(t) + C, \qquad (2.2)$$

where C is an arbitrary constant. We define the ***Cauchy integral*** of f by

$$\int_r^s f(t)\,\Delta t = F(s) - F(r), \quad r, s \in \mathbb{T}. \tag{2.3}$$

Definition 2.9. We say that a function $p\colon \mathbb{T} \to \mathbb{R}$ is ***regressive*** provided

$$1 + \mu(t)p(t) \neq 0, \qquad \textit{for all } t \in \mathbb{T}.$$

The set of all regressive functions $f\colon \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Definition 2.10(The Generalized Exponential Function).

If $p \in \mathcal{R}$, then we define the exponential function$e_p(t,s)$by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(t)}\big(p(\tau)\big)\Delta\tau\right), \qquad for\ s, t \in \mathbb{T},$$

where

$$\xi_{\mu(s)}\big(p(s)\big) = \begin{cases} \dfrac{1}{\mu(s)} log(|1 + \mu(s)p(s)| + i\,Arg(1 + \mu(s)p(s))), & for\ \mu(s) > 0 \\ p(s) & \mu(s) = 0 \end{cases}$$

For more details about the exponential function see [2]

Definition 2.11. [5] We say that the n^{th}order dynamic equation

$$x^{\Delta^n}(t) = F\big(t, x^{\Delta^{n-1}}, .. x^{\Delta}, x\big), \ \ t \in \mathbb{T}. \tag{2.4}$$

has Hyers-Ulam stability on $\mathbb{T}$ if for every $\varepsilon > 0$ and $u \in C_{rd}^n(\mathbb{T}, \mathbb{X})$ which satisfies

$$\left\| u^{\Delta^n}(t) - F(t, u^{\Delta^{n-1}}(t) \dots, u^{\Delta}(t), u) \right\| < \varepsilon, \qquad t \in \mathbb{T},$$

there exists a solution x of (2.4) such that:

$$\|u(t) - x(t)\| < L\varepsilon \,, \qquad t \in \mathbb{T} \text{ for some } L > 0.$$

Definition 2.12[10] If for each function y satisfying

$$\left| y(x) - \int_a^x f\big(x, \tau, y(\tau)\big)\, d\tau \right| \leq \psi(x) \;\; (-\infty < a \leq x \leq b < +\infty) \;\; (2.5)$$

where ψ is a non-negative function, there exists a solution y_0 of the Volterra integral equation (1.1) and a constant C_1 > 0 independent of y and y_0 such that

$$|y(x) - y_0(x)| \leq C_1\, \psi(x), \qquad (2.6)$$

for all x, then we say that the integral equation (1.1) has the Hyers–Ulam–Rassias stability.

Definition 1.13. In the particular case of Definition 1.12 when ψ is just a constant function in the above inequalities, we say that the integral equation (1.1) has Hyers–Ulam stability.

3. Hyers-Ulam-Rassias For Volterra Integral Equations on Time scales

In this section we investigate Hyers-Ulam-Rassias stability of the Volterra integral equations on time scales of the form

$$y(t) = \int_{t_0}^{t} f\big(t, s, y(s)\big)\, \Delta s, \qquad t \in \mathbb{T} \;, \qquad (3.1)$$

Theorem 3.1. *Let* k , L *are positive constants and assume that* $f:\mathbb{T}\times\mathbb{T}\times\mathbb{X}\rightarrow\mathbb{X}$ *is a continuous function which additionally satisfies the Lipschitz condition*

$$\|f(t,s,y) - f(t,s,z)\| \le L \|y - z\| \tag{3.2}$$

for any $t,s\in\mathbb{T}$ *and all* $y,z \in C_{rd}(\mathbb{T},\mathbb{X})$
If a function $g\in C_{rd}(\mathbb{T},\mathbb{X})$ *satisfies*

$$\left\|g(t)-\int_{t_o}^{t} f(t,s,g(s))\,\Delta s\right\| \le \theta(t) \ ,t\in\mathbb{T} \tag{3.3}$$

where $\theta\in C_{rd}(\mathbb{T},\mathbb{R}^{+})$ *with*

$$\int_{t_o}^{t} e_L(t,\sigma(s))\,\theta(s)\,\Delta s\le k\,\theta(t),\ \ k>0 \quad t\in\mathbb{T} \tag{3.4}$$

then there exists a unique function $u\in C_{rd}(\mathbb{T},\mathbb{X})$ *of the Equation* (3.1), such that

$$\|g(t)-u(t)\| \le (1+Lk\ )\,\theta(t), \quad t\in\mathbb{T} \tag{3.5}$$

Proof. Given $\theta\in C_{rd}(\mathbb{T},\mathbb{R}^{+})$ and suppose $g\in C_{rd}(\mathbb{T},\mathbb{X})$ satisfies

$$\left\|g(t)-\int_{t_o}^{t} f(t,s,g(s))\,\Delta s\right\| \le \theta(t), \qquad t\in\mathbb{T} \tag{3.6}$$

Set

$$\ell(t) = g(t) - \int_{t_o}^{t} f(t, s, g(s))\,\Delta s, \qquad t \in \mathbb{T}. \tag{3.7}$$

So $\|\ell(t)\| \leq \theta(t)$ (3.8)

Let $u \in C_{rd}(\mathbb{T}, \mathbb{X})$ be the unique solution of the Volterra equation (3.1) .

Then

$$\|g(t) - u(t)\| =$$

$$= \left\| \ell(t) + \int_{t_o}^{t} f(t, s, g(s))\,\Delta s - \int_{t_o}^{t} f(t, s, u(s))\,\Delta s \right\|$$

$$\leq \|\ell(t)\| + \left\| \int_{t_o}^{t} [f(t, s, g(s)) - f(t, s, u(s))] \Delta s \right\|$$

$$\leq \|\ell(t)\| + \int_{t_o}^{t} \|[f(t, s, g(s)) - f(t, s, u(s))]\|\,\Delta s$$

$$\leq \|\ell(t)\| + L \int_{t_o}^{t} \|g(s) - u(s)\|\,\Delta s \quad \text{(from (3.2))}$$

By using Gronwall"s Inequality in [8] , we have that

$$\|g(t) - u(t)\| \leq \theta(s) + \int_{t_o}^{t} e_L(t, \sigma(s))\, \theta(s)\, L\, \Delta s$$

$$\leq \theta(s) + L \int_{t_o}^{t} e_L(t, \sigma(s))\, \theta(s)\, \Delta s$$

$$\leq \theta(t) + Lk\, \theta(t)$$

$$\leq (1 + Lk\,)\theta(t)$$

The proof is complete. ■

References:

[1] M. Akkouchi - Hyers-Ulam-Rassias stability of nonlinear Volterra integral,(2003), no. 1, 136-146.

[2] Alaa E. Hamza, Moneer Al-Qubaty, On The Exponential Operator Functions on Time Scales, Advances in Dynamical Systems and Applications, 7(1) (2012), 57-80.

[3] Alaa E. Hamza, Nesreen A.Yaseen, Hyers-Ulam stability of first order abstract linear dynamic equations on time scales, Advances in mathematics, 7(1) (2014), 1095-1104.

[4] Alaa E. Hamza, Nesreen A.Yaseen, Hyers-Ulam stability of abstract second order linear dynamic equations on time scales , Int. Journal of Math. Analysis, (8)(2014), 1421-1432.

[5] D. R. Anderson, B. Gates and D. Heuer, Hyers-Ulam stability of second -0rder linear dynamic equations on time scales, Communications in applied Analysis, 16 (3)(2012), 281-292.

[6] B. Aulbach and S.Hilger, Linear Dynamic Processes with Inhomogenous Time Scale, Nonlinear Dynamic and Quantum Dynamical Systems (Gaussing,(1990)

Math. Res. Akademic-Verlag, Berlin, 59(1990), 9-20.

[7] B. Aulbach and S.Hilger, AUnified Approach to Continuous and Discrete Dina- mic, Qualitative Theory of differential Equations (Szeged,1988), Colloq. Math. Soc. Janos Bolyai. North Holland, Ameserdam, (1990),37-56.

[8] M. Bohner, A. Peterson, Dynamic Equations on Time Scale: An introduction with Applications, Birkhauser, Birkhauser, Boston. Basel. Berlin, (2001).

[9] T.A. Burton, Volterra Integral and Differential Equations, Mathematics in Science and Engineering 2002, Elsevier, Amsterdam, 2005.

[10] L. P. Castro. and A. Ramos ,Hyers-Ulam-Rassias stability for a class of non- linear Volterra integral equations, Banach J. Math. Anal. 3 (1)(2009), 36– 43.

[11] C. Constanda, M.E. Pérez , Integral Methods in Science and Engineering, Spr-inger New York Dordrecht, Heidelberg London, (2010), DOI 10.1007/978.

[12] C. Corduneanu, Principles of Differential and Integral Equations, Chelsea,NewYork, (1988).

[13] M. Gachpazan, O.Baghani, Hyers-Ulam stability of nonlinear integral equa- tion *Fixed Point Theory and Applicaions*, (2010), doi:10.1155/2010/

927640.

[14] M. Gachpazan, O.Baghani, Hyers-Ulamstability of Volterra integral equation. Journal of Nonlinear Analysis and its Applications, (2)(2010),19-25 .

[15] R. Ger, and C. Alsina, On some inequalities and stability results related to the exponential function, J. Inequal. Appl.2 (1998), 373-380.

[16] S. Hilger, Ein MaBkettenkalkul mit Anwendungauf Zentrumsmannigfaltig- keiten, PH .D thesis, Universitat Wurzburg, (1988).

[17] D. H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. U.S.A, 27 (1941), 222-224.

[18] S.-M, Jung: A fixed point approach to the stability of a Volterra integral equation. Fixed Point Theory and Applications, (2007).

[19] V. Lakshmikantham and M.R.M. Rao, Theory of Integro differential Equation Stability and Control: Theory, Methods and Applications 1, Gordon and Bre- ach Publ., Philadelphia, (1995).

[20] Th. M. Rassias, On the stability of linear mapping in Banach spaces, Proc.Amer. Math. Soc. 72(1978), 297-300.

[21] J. M. Rassias, special editor-inchief, Special Issue on Leonhard Paul Eulers Functional Equations and Inequalities, International J. Appl. Math. & Stat.,7(2007).

[22] S. M. Ulam, A Collection of the Mathematical Problems, Interscience, New York, (1960).